

A CLASS OF $N = 1$ DUAL STRING PAIRS AND ITS MODULAR SUPERPOTENTIAL

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ABSTRACT

We compare the $N = 1$ F-theory compactification of Donagi, Grassi and Witten with modular superpotential - and some closely related models - to dual heterotic models. We read off the F -theory spectrum from the cohomology of the fourfold and discuss on the heterotic side the gauge bundle moduli sector (including the spectral surface) and the necessary fivebranes. Then we consider the $N = 1$ superpotential and show how a heterotic superpotential matching the F-theory computation is built up by worldsheet instantons. Finally we discuss how the original modular superpotential should be corrected by an additional modular correction factor, which on the F -theory side matches nicely with a ‘curve counting function’ for the del Pezzo surface. On the heterotic side we derive the same factor demanding correct T -duality transformation properties of the superpotential.

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1 Introduction

During the last two years accumulating and convincing evidence for the $N = 2$ string-string duality between heterotic string on $K3 \times T^2$ and the type IIA string on a corresponding Calabi-Yau three-fold was obtained [1, 2, 3, 4, 5, 6, 7]. The $N = 2$ string-string duality can be, at least heuristically, derived by considering as a starting point the $N = 4$ string-string duality [8] between the heterotic string on T^6 and the type IIA string on $K3 \times T^2$ and then performing a kind of orbifoldization which breaks half of the space-time supersymmetry. The information about the (perturbative) heterotic spectra is encoded by a particular choice of a gauge bundle over $K3$, which has to be matched by the topological data of the type IIA $K3$ -fibration. Furthermore, non-perturbative states can emerge when considering various types of (compactified) branes on both sides.

The same type of techniques can be also applied when constructing dual string pairs with $N = 1$ supersymmetry in four dimensions. Namely first, dual $N = 1$ string pairs were obtained by orbifolding already known $N = 2$ dual pairs [9]. More recently there has begun a corresponding investigation of the $N = 1$ duality between the heterotic string on a Calabi-Yau three-fold - assumed to be elliptically fibered over a complex surface - together with a certain bundle embedded in the gauge bundle, and F-theory [10] on a Calabi-Yau four-fold, which is assumed to be $K3$ fibered over the same surface, i.e. one is adiabatically extending the corresponding eight-dimensional duality [11, 12, 13, 14, 15]. In a certain sense we will combine both techniques in this paper.

Now besides matching the spectra and enhanced gauge symmetries there were also refined checks of the $N = 2$ duality, where a holomorphic quantity, the prepotential, was compared on the heterotic side and on the type II side. There, one was restricted to weak coupling on the heterotic side, where - because of T-duality - modular functions played an important role, whereas for the corresponding quantity on the type II side a world-sheet instanton sum played the dominant role.

Investigating $N = 1$ dual string pairs, possible checks, that go beyond matching the spectrum, involve the comparison of $N = 1$ effective interactions which are determined by holomorphic quantities, namely the superpotential or the gauge kinetic function. In this paper we will make a duality match between the superpotential, generated by perturbative effects on the heterotic side, and on the F -theory side a certain sum over geometrical objects, which produce instanton contributions (five-branes in the M-theory set-up wrapped over certain six-cycles resp. the type IIB three-branes over corresponding four cycles [11]). We will consider models, where the F -theory four-folds are $K3$ fibrations over dP of Euler number $\chi = 12 \cdot 24$. (dP stands for the del Pezzo surface B_9 , the projective plane blown up in the nine intersection points of two cubics.) Moreover, these four-folds are elliptically fibered over $dP \times P^1$ (the non Calabi-Yau three-fold base of type IIB with varying dilaton). The nice thing in this class of models [12] is that, like as for the $N = 2$ prepotential, we will get for the superpotential a modular function and furthermore the six-cycles reduce effectively again to rational curves in the threefold $dP \times P^1$; namely the relevant four-cycles are of the form ‘section $\times P^1$ ’, where the first factor describes a section of the elliptic fibration of the del Pezzo over its own rational base P^1_{dP} . The Calabi-Yau threefold of all the heterotic dual models we are considering is given by an elliptic fibration over dP , which has Hodge numbers $h^{(1,1)} = h^{(2,1)} = 19$, denoted here as $CY^{(19,19)}$ [16]. The different F -theory compactification just correspond

to different choices of heterotic gauge bundles over $CY^{(19,19)}$. We show that the rational curves on the heterotic side reproduce the modular F -theory superpotential.

Our paper is organized as follows. After discussing the four-fold X^4 which was used in [12] to obtain the modular superpotential, we will consider in chapter two closely related F -theory compactifications which can be obtained from $N = 2$ supersymmetric F -theory compactifications by a Z_2 modding. Specifically, the $N = 2$ parent fourfolds will be either given by $CY^{(3,243)} \times T^2$ (equivalent to IIA compactification on $CY^{(3,243)}$), or by $K3 \times K3$. In the first type of models the non-vanishing Euler number of the $N = 1$ fourfold X^4 and hence the twelve three-branes emerge by the \mathbf{Z}_2 modding; at the same time the visible dP emerges from a \mathbf{Z}_2 modding of $P^1 \times T^2$; therefore we call this model of ‘*emergence*’ type. The second class of models, in which one first goes to eight dimensions and then to four dimensions on $K3$, we call ‘*reduction*’ type since the Euler number and so the number of 24 three-branes is reduced by half due to the \mathbf{Z}_2 modding. Similarly the visible dP is reduced from the $K3$ by the modding procedure. On the heterotic side these two different types of F -theory compactification will correspond to different choices of gauge bundles with, however, same internal Calabi-Yau threefold $CY^{(19,19)}$. One can regard the different heterotic gauge bundles also having either a six-dimensional or eight-dimensional origin, respectively. We will also discuss the spectral surface in the bundle moduli sector and the emergence/reduction of the corresponding twelve heterotic fivebranes.

In chapter three we will discuss how to match the F -theory and heterotic superpotentials. In this context the question arises of how to correct the superpotential of [12] by an η power denominator, which is derived first via mirror symmetry and then independently via a heterotic orbifold computation using the modular weight arguments based on the fact that the superpotential has to balance the Kähler potential with respect to T -duality transformations [17]. We also observe the occurrence of a second E_8 theta-function.

For convenience of the reader some facts on the del Pezzo surface $dP = \left[\begin{smallmatrix} P^2 \\ P^1 \end{smallmatrix} \middle| \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right]$, the Calabi-Yau space $CY^{19,19} = \left[\begin{smallmatrix} P^2 \\ P^1 \\ P^2 \end{smallmatrix} \middle| \begin{smallmatrix} 3 & 0 \\ 1 & 1 \\ 0 & 3 \end{smallmatrix} \right]$ and the Calabi-Yau fourfold X^4 of [12], which are assumed to be known throughout the main body of the paper, are collected in the appendix.

2 F -theory over $dP \times P^1$ and dual heterotic models on the $CY^{19,19}$

We start with a heuristic comparison of the incomplete data consisting of the threefold base B^3 of the elliptically fibered Calabi-Yau fourfold on the F -theory side and the heterotic Calabi-Yau (without bundle). Then we go on and describe (section 2.1) Calabi-Yau fourfolds elliptically fibered over the base $B^3 = dP \times P^1$ and complete also (section 2.2) the specification of the (0,2) heterotic Calabi-Yau model which involves additionally the choice of a stable, holomorphic vector bundle to be embedded in the gauge bundle.

The Calabi-Yau fourfold X^4 (we call it model A) for F -theory compactification used in

[12] giving a modular superpotential is defined as

$$X_A^4 = \left[\begin{array}{c|cc} P_x^2 & 3 & 0 \\ P_y^1 & 1 & 1 \\ P_z^1 & 0 & 2 \\ P_w^2 & 0 & 3 \end{array} \right] \quad (2.1)$$

representing a complete intersection in the product of the projective spaces listed on the left given by two equations of the listed multidegrees.

Let us first discuss the fibration structure of the fourfold X^4 . Because of the two plane cubics occuring here X^4 can be seen in two ways as being elliptically fibered over a threefold base. Especially the elliptic F-theory fibration by the T_w^2 of the last row over $B = dP_{x,y} \times P_z^1$ is characteristic for model A. Obviously X_A^4 is a $K3 = \left[\begin{array}{c|c} P_z^1 & 2 \\ P_w^2 & 3 \end{array} \right]$ fibration over the mentioned del Pezzo (more precisely, the $K3$ varies over the base P_y^1 of the dP , but not over its elliptic fibre T_x^2 . Its Picard number is $\rho = 2$, leaving 18 deformations). If you fibre the fourfold over P_y^1 , the threefold fibre is given by $T_x^2 \times K3$; this exhibits the total space as the fibre product $X_A^4 = dP \times_{P_y^1} \mathcal{B}_A$ of Euler number $\chi = 12 \cdot 24$ with the non-CY three-fold $\mathcal{B}_A = \left[\begin{array}{c|c} P_y^1 & 1 \\ P_z^1 & 2 \\ P_w^2 & 3 \end{array} \right]$, which is fibered by the mentioned K3 over P_y^1 .

Let us now determine the heterotic Calabi-Yau 3-fold which is dual to X_A^4 or better to F-theory over $dP \times P^1$. As the X^4 models lead to type IIB on $B = dP \times P_z^1$ one can use the duality in eight dimensions between type IIB on P_z^1 (resp. - taking into account the additional information provided by the 7-brane locations/degenerate elliptic fibers - between F-theory on the $K3 = \left[\begin{array}{c|c} P_z^1 & 2 \\ P_w^2 & 3 \end{array} \right]$) and the heterotic string on T_{het}^2 and then spread it out over dP to four dimensions. The volume of P_z^1 will correspond to the heterotic dilaton. This leads to the heterotic string on a Calabi-Yau threefold, which is elliptically fibered over del Pezzo, for which purpose the $CY^{(19,19)} = dP \times_{P^1} dP$ presents itself naturally.² Note, that besides being an elliptic fibration, X_A^4 itself is also a fibration of the $CY_{x,y,w}^{19,19}$ over P_z^1 .

Let us remark that we will have some variability for the model building to follow: the only feature of X_A^4 which matters for the modular superpotential is that it is a fibre product $dP \times_{P^1} \mathcal{B}$ built with a threefold \mathcal{B} of $h^{1,0}(\mathcal{B}) = h^{2,0}(\mathcal{B}) = h^{3,0}(\mathcal{B}) = 0$, which is K3 fibered over P_y^1 and even elliptically fibered over $F_0 = P_y^1 \times P_z^1$, so that the IIB base is $dP \times P_z^1$.

2.1 F-theory side

Before we enter the discussion about specific F-theory compactifications, let us consider the question of determination of the spectrum in somewhat more general terms (we will

²Note that besides the already visible dP the second one can be argued heuristically to arise as follows: if one considers for the moment only this sector, i.e. (in 6D say) the variation of T_{het}^2 over the base P_y^1 of the visible del Pezzo, then this should match the variation of the $K3_{12-8}$ over P_y^1 on the F-theory side, i.e. the K3 fibered (non CY) threefold $\mathcal{B}_A = \left[\begin{array}{c|c} P_y^1 & 1 \\ P_z^1 & 2 \\ P_w^2 & 3 \end{array} \right]$; but this space can be pulled back quadratically in the base P_y^1 to a CY just like the corresponding pull back would lead on the heterotic side from del Pezzo to a $K3 = \left[\begin{array}{c|c} P^1 & 2 \\ P^2 & 3 \end{array} \right]$ appropriate to correspond to a CY.

consider the brane sector later). Besides the Kähler and complex structure parameters related to $h^{1,1} - 2$ (not counting the unphysical zero-size F-theory elliptic fibre as well as not counting the class corresponding to the heterotic dilaton) and $h^{3,1}$ respectively, we have to take into account the contribution of $h^{2,1}$ giving in total $h^{1,1} - 2 + h^{2,1} + h^{3,1}$ parameters which equals $\frac{\chi}{6} - 10 + 2h^{2,1}$ according to [18]. All these contributions divide themselves between chiral and vector multiplets (just as in the analogous 6D $N = 2$ case [16]) according to whether or not they come from the threefold base B^3 of the F-theory elliptic fibration. So we expect for the rank v of the $N = 1$ vector multiplets (unspecified hodge numbers relate to X^4) (cf. also [19])

$$v = h^{1,1} - h^{1,1}(B^3) - 1 + h^{2,1}(B^3) \quad (2.2)$$

and for the number c of $N = 1$ neutral chiral (resp. anti-chiral) multiplets

$$\begin{aligned} c &= h^{1,1}(B^3) - 1 + h^{2,1} - h^{2,1}(B^3) + h^{3,1} \\ &= h^{1,1} - 2 + h^{2,1} + h^{3,1} - v = \frac{\chi}{6} - 10 + 2h^{2,1} - v. \end{aligned} \quad (2.3)$$

Now note that as our models are fibre products of del Pezzo and a K3 fibered threefold \mathcal{B} one has with $h^{(1,1)} = 10 + \rho = 12$, where ρ denotes the Picard number of the K3 of the threefold, that

$$v = \rho - 2 + h^{2,1}(B^3). \quad (2.4)$$

Now in constructing specific F -theory fourfolds, we make use of the fact that the heterotic $CY^{19,19}$ is a \mathbf{Z}_2 orbifold of the space $K3 \times T^2$, which represents the geometric starting point for a heterotic $N = 2$ compactification. As explained in the appendix A.2 the dP 's in the $CY^{(19,19)}$ arise in two different ways, namely either by ‘*emergence*’ or by ‘*reduction*’ from $K3 \times T^2$ (the differences will show up again in two different choices of heterotic gauge bundles). Therefore on the F -theory side again two possible Z_2 moddings present themselves naturally. First X^4 might be obtained by modding out the corresponding \mathbf{Z}_2

involution on $T^2 \times CY = \left[\begin{array}{c} P_x^2 \\ P_y^1 \\ P_z^1 \\ P_w^2 \end{array} \left| \begin{array}{cc} 3 & 0 \\ 0 & 2 \\ 0 & 2 \\ 0 & 3 \end{array} \right. \right]$ (model A), i.e. the model is a Z_2 orbifold of the type IIA string on the $CY = \left[\begin{array}{c} P^1 \\ P^1 \\ P^2 \end{array} \left| \begin{array}{c} 2 \\ 2 \\ 3 \end{array} \right. \right]$. On the other hand, X^4 can also be obtained by

modding out \mathbf{Z}_2 involutions on $K3 \times K3 = \left[\begin{array}{c} P_x^2 \\ P_y^1 \\ P_z^1 \\ P_w^2 \end{array} \left| \begin{array}{cc} 3 & 0 \\ 2 & 0 \\ 0 & 2 \\ 0 & 3 \end{array} \right. \right]$, which we call model C (we leave out model B to avoid confusion in notations).

Let us first consider the ‘*emergence*’ type of models, where the Euler number, $\chi = 12 \cdot 24$, the three-branes, $n_3 = 12$, and also the dP emerge after the Z_2 modding. Note that the Calabi-Yau $\left[\begin{array}{c} P^1 \\ P^1 \\ P^2 \end{array} \left| \begin{array}{c} 2 \\ 2 \\ 3 \end{array} \right. \right]$ is elliptic over F_0 and of Hodge numbers $h^{(1,1)} = 3$ as coming from the factors of the ambient space and $h^{(2,1)} = 3 \times 3 \times 10 - (3 + 3 + 8) - 1 = 75$ (for more on this crucial number 75 cf. appendix A.3); this does not satisfy the six-dimensional anomaly condition for an elliptically fibered Calabi-Yau threefold of $h^{(1,1)} = 3$, which would be forced to have $h^{(2,1)} = 243$. But note that this CY, as well as the fourfold X_A^4 , does not have a section (as its K3 already only has a trisection: the line in P^2). So we will postpone the discussion of this model to the appendix. It may still exist as a genuine (not modded from a $N = 2$ situation) $N = 1$ model.

So we will use instead of $CY^{(3,75)}$ the ‘consistent’ $CY^{(3,243)}$ (of equation $y^2 = x^3 - f_{8,8}x - g_{12,12}$ with $h^{(2,1)} = 9^2 + 13^2 - (3 + 3 + 1)$), i.e. we actually will consider the model A'

$$X_{A'}^4 = (T^2 \times CY^{(3,243)})/\mathbf{Z}_2 = dP \times_{P_y^1} \mathcal{B}_{A'}, \quad (2.5)$$

where $\mathcal{B}_{A'}$ is the appropriately \mathbf{Z}_2 modded $CY^{(3,243)}$, i.e. $\mathcal{B}_{A'} : y^2 = x^3 - f_{4,8}x - g_{6,12}$ with $5 \times 9 + 7 \times 13 - (3 + 3 + 1) = 129$ deformations,³ which then gives $h^{(3,1)}(X_{A'}^4) = 8 + 3 + 129 = 140 = 5 \times 28$ and using $40 = \frac{x}{6} - 8 = h^{1,1} - h^{2,1} + h^{3,1}$ (cf. [18]) and $h^{1,1} = 12$ that $h^{(2,1)}(X_{A'}^4) = 112 = 4 \times 28$.

Having computed all the relevant Hodge numbers we can easily determine the spectrum for model A' from eqs.(2.2) and (2.3). Recall that $h^{2,1}(B^3 = dP \times P^1) = 0$ and $\rho = 2$, we finally derive that $v = 0$ and $(\chi = 288)$

$$c = 38 + 2h^{2,1} = 262. \quad (2.6)$$

Now let us come to the ‘reduction’ type models, where the Euler number, $\chi = 12 \cdot 24$, and the three-branes, $n_3 = 12$, are obtained via reduction by the \mathbf{Z}_2 modding. Moreover here $K3$ is reduced to dP . Using $X_{C,C'}^4 = (K3 \times K3)/\mathbf{Z}_2$ with the $(10, 8, 0)$ involution (cf. [20]) in the (say) first $K3_{8-4}$ giving the visible del Pezzo we consider the two options $(10, 10, 0)$ resp. $(10, 8, 0)$ concerning the involution of Nikulin type (r_2, a_2, δ) in the second $K3_{12-8}$. Note that these spaces are still fibre products of del Pezzo (coming from $K3_{8-4}$) and a threefold ($K3_{12-8}$ fibered over the P^1 base of del Pezzo) over the P^1 base of del Pezzo. One gets $h^{(1,1)} = 10 + r_2 + \alpha$, where $r_2 = \rho = 10$ and α denotes the contribution from resolving the singularities caused by the fixed locus in case C' . Furthermore $h_C^{(2,1)} = 0$ resp. $h_{C'}^{(2,1)} = 8$ as this odd cohomology can come only from the fix locus: the two $K3$ lead to two base P^1 and two elliptic directions and in the case C' one has $2 \times 2 = 4$ ordinary \mathbf{Z}_2 singularities (in the ‘plane’ built by the two P^1 directions) ‘multiplied’ by the two elliptic directions, which leads for each of the four loci to $P_{\text{res}}^1 \times E_{\text{vis}} \times E_{11,12}$ of respectively four $h^{(2,1)}$ classes (by wedging in the mentioned order the classes in $(h^{1,1}, h^{1,0}, h^{0,0}), (h^{1,1}, h^{0,0}, h^{1,0}), (h^{0,0}, h^{1,1}, h^{1,0}), (h^{0,0}, h^{1,0}, h^{1,1})$) of which only the first two lead to new cohomology in X^4 . So one gets that for $((10, 8, 0), (10, 10, 0))$ of base $dP \times P^1 = dP \times_{P_y^1} F_0$, i.e. model C , $h^{(1,1)} = 20$, $h^{(2,1)} = 0$, $h^{(3,1)} = 20$, so $v = 8$, corresponding to a rank 8 gauge group, and $c = 38 - 8 = 30$.

For model C' with $((10, 8, 0), (10, 8, 0))$ involution of base $dP \times_{P_y^1} Bl_4(F_0)$ we derive that $h^{(1,1)} = 24$, $h^{(2,1)} = 8$, $h^{(3,1)} = 24$ so $v = 8 + 4$, $c = 38 + 2 \cdot 8 - (8 + 4) = 42 = 30 + 12$.⁴ Note that the newly introduced classes in the Bl_4 process do not lead to a further divisor contributing to the superpotential as $\chi_{\text{ar}}(P_{\text{res}}^1 \times E_{\text{vis}} \times E_{11,12}) = 0 \neq 1$ because this divisor has $h^{3,0} = 0$, $h^{2,0} = 1$, $h^{1,0} = 2$.

³whereas $h^{2,1}(\mathcal{B}_{A'}) = 112$ as we will see (compare the corresponding difference of $\sharp_{\text{def}} \mathcal{B}_A = 45$, $h^{2,1}(\mathcal{B}_A) = 28$ in the A model (cf. A.3)); note that the number of complex deformations $\sharp_{\text{def}} \mathcal{B}$ of the non-Calabi-Yau space \mathcal{B} differs from $h^{2,1}(\mathcal{B})$ by 17, resp. $\sharp_{\text{def}} K3_{12-8} - 1$ in general, which equivalently makes possible to have the identity $h^{2,1}(\mathcal{B}) = h^{2,1}(X^4)$ as $\sharp_{\text{def}} \mathcal{B} = \sharp_{\text{def}} K3 - 1 + h^{2,1}(\mathcal{B}) = 20 - \rho - 1 + h^{2,1}(\mathcal{B}) \leftrightarrow h^{3,1} = 8 + 3 + \sharp_{\text{def}} \mathcal{B} = 30 - \rho + h^{2,1}(\mathcal{B}) \leftrightarrow h^{2,1} = h^{1,1} + h^{3,1} - 40 = h^{2,1}(\mathcal{B})$

⁴We expect that the 4 vector multiplets and 12 chiral multiplets, which come in addition compared to model C , are non-perturbative on the heterotic side, since they arise from the blowing up of the type II base, like four additional heterotic fivebranes (wrapping now T_{56}^2 instead of $T_{9,10}^2$) with their $12 = 4 \cdot 3$ parameters.

2.2 Heterotic side

The nice thing about the $CY^{(19,19)}$ is of course that it is a \mathbf{Z}_2 orbifold of $K3 \times T^2$. Now a $N = 2$ heterotic string model on $K3 \times T^2$ is specified by a choice of gauge bundle in $E_8 \times E_8$. If we consider a $(n_1, n_2; n_5)$ situation, where besides an $SU(2)$ gauge bundle with instanton numbers (n_1, n_2) in $E_8 \times E_8$ also n_5 heterotic fivebranes are turned on, then the anomaly cancellation condition in six dimension reads $n_1 + n_2 + n_5 = 24$. Specifically we are considering the complete Higgsed situation which is equivalent to start from $E_8 \times E_8$ instantons. Models A' and C will represent the extreme choices of numbers of heterotic five branes, namely $n_5 = 0$ or $n_5 = 24$ respectively,

Specifically, the $N = 2$ parent of the heterotic dual of model A' is characterized by $n_1 = n_2 = 12$, $n_5 = 0$. After the \mathbf{Z}_2 modding, breaking $N = 2$ to $N = 1$, the number of moduli is $h_{het}^{(1,1)} + h_{het}^{(2,1)} + x$, where x denotes the number of heterotic gauge bundle parameters, i.e. here the number of surviving instanton moduli. Note that the absence of fivebranes, $n_5 = 0$, in the $(12,12;0)$ situation is consistent with the absence of 3-branes in the $N = 2$ F -theory, since the Euler number of $CY^{(3,243)} \times T^2$ is zero. As in this case there are no preexistent fivebranes let us see how after going to $N = 1$ the (necessary to match the $12 = \frac{X}{24}$ F -theory threebranes) heterotic fivebranes arise by ‘*emergence*’. Namely one has to fulfill $c_2(V_1) + c_2(V_2) + n_5[f] = c_2(CY)$, where a number n_5 of fivebranes wrapping the elliptic fibre f of the CY over its base B is allowed (and required). The evaluation $c_2(CY) \cdot J_1 = 3 \cdot 3 + 3 \cdot 9 = 36$ then gives (via the relation of J_1 with f) the relation $n_5 = 12$ (cf. also [21],[22] and A.2).

Let us now come to the discussion about the heterotic spectrum, in particular the question about the heterotic gauge bundle. Since the gauge group was completely broken by the $(12,12)$ instantons we could expect therefore that after the Z_2 modding (which acts freely on the original six-torus, see appendix A.2) there are no $N = 1$ vector multiplets, in agreement with the F -theory prediction $v = 0$. Next consider the surviving scalar fields after the Z_2 twist. Recall that the number of $N = 2$ hypermultiplets was given by the number of $K3$ deformations plus the quaternionic dimension of the instanton moduli space,

$$m_{inst} = \dim_{quat}(\mathcal{M}_{12}^{inst} \times \mathcal{M}_{12}^{inst}) = 2(c_2(E_8) \times 12 - 248) = 2 \times 112. \quad (2.7)$$

Hence in the $N = 2$ situation we count $H = 20_{K3_{het}} + m_{inst}^{N=2} = 244$ hypermultiplets. After Z_2 modding we get as number of chiral deformations first the number of Kähler and complex structure parameters of $CY^{(19,19)}$, i.e. $h_{het}^{1,1} + h_{het}^{2,1} = 19 + 19$. Second, of each of the $N = 2$ instanton hypermultiplets there survives one of their two chiral multiplets. So in total

$$c = h_{het}^{1,1} + h_{het}^{2,1} + x \quad (2.8)$$

$$= 19 + 19 + m_{inst} \quad (2.9)$$

$$= 38 + 224. \quad (2.10)$$

This number matches⁵ with the corresponding F -theory prediction, i.e. $x = 2h^{2,1}(X^4)$.

⁵Note that in the setup of such a \mathbf{Z}_2 modding of type IIA on CY^3 (i.e. model A') one has with a number of $\sharp H = h^{2,1}(CY^3) + 1$ hypermultiplets, $e_{CY^3} = 2e_B - 2 \cdot 24$ by the ramified covering and $e_{CY^3} = 2(3 - h^{2,1}(CY^3))$, $e_B = 2 + 2(3 - h^{2,1}(B))$ as $h^{3,0}(B) = 0$ the conclusion $2h^{2,1}(B) = m_{inst}^{N=2}$ (so $h^{2,1}(B) = 112$) which gives with $h^{2,1}(X^4) = h^{2,1}(B)$ the match $2h^{2,1}(X^4) = m_{inst}^{N=2} = x$.

Let us indicate from a somewhat more general perspective that here - in the gauge bundle moduli sector - indeed the spectrum of the modded $N = 2$ parent model is simply the modded spectrum of the $N = 2$ model. For this note that in (leaving out the intermediate step over $X_{11}(j)$)

$$\begin{array}{ccc} K3_{10-6} \times T_{6-4}^2 & \longrightarrow & CY^{19,19} \\ \downarrow -ell_{K3} & & \downarrow -ell_{dP_{10-6}} \\ P_{K3}^1 \times T_{6-4}^2 & \longrightarrow & dP_{8-4} \end{array} \quad (2.11)$$

a bundle V over $CY^{19,19}$ - to be considered as being modded from the $N = 2$ situation on the left hand side - has by consistency to pullback to a bundle 'living' (varying) purely in the $K3$ -sector, i.e. V must not vary along $ell_{dP_{8-4}}$. So the spectral surface C_{spec}^2 (at first over X_{11} in the intermediate step, say) is double covered (with branching only in codimension two) by $C_{spec}^1 \times T_{6-4}^2$ with C_{spec}^1 the spectral curve of the $N = 2$ parent model on the left. So counting the deformations in the spirit of [22] one finds that again by $h^{2,0}(C_{spec}^2) = h^{2,0}(C_{spec}^1 \times T^2) = h^{1,0}(C_{spec}^1)$ the relevant number simply persists.

Furthermore this sheds in our special case also light on a conjectured relation [21] between $h^{2,1}(X^4)$ and $h^{1,0}(C_{spec}^2)$: $h^{2,1}(X^4) = h^{2,1}(\mathcal{B}) = m_{inst}^{N=2}$ was the relevant number of deformations on the $N = 2$ level, i.e. $h^{1,0}(C_{spec}^1)$, and on the other hand $h^{1,0}(C_{spec}^2) = h^{1,0}(C_{spec}^1 \times T^2) = h^{1,0}(C_{spec}^1) + 1$. Concerning the also conjectured relation between the abelian varieties, the Albanese $Alb(C_{spec}^2)$ and the intermediate Jacobian $Jac_{intmed}^{(2,1)}(X^4)$, note that in our setup the first is now related to $Jac(C_{spec}^1)$, whose relation with $Jac_{intmed}(\mathcal{B})$ (related to $Jac_{intmed}^{(2,1)}(X^4)$, extending their dimensional identity; $Jac_{intmed}(\mathcal{B})$ occurs here as capturing the relevant part of $Jac_{intmed}(CY_F^3)$) should then be part of a corresponding $N = 2$ relation.

Let us, more briefly, also discuss the heterotic duals of the F -theory models C and C' . For model C we need on the level of the $N = 2$ parent model $24 = \frac{\chi_{K3 \times K3}}{24}$ threebranes on the F -theory side; so we need $n_5 = 24$ on the heterotic side, and we do not turn on any gauge bundle in the dual heterotic model, i.e. $(n_1, n_2, n_5) = (0, 0; 24)$. (The gauge group $E_8 \times E_8$ would remain unbroken in six dimensions.) After the \mathbf{Z}_2 modding the 12 heterotic fivebranes arise by 'reduction' from the 24 fivebranes in the $N = 2$ situation. Remember that for models C, C' we obtained $h^{(2,1)}(X_C^4) = 0$, $h^{(2,1)}(X_{C'}^4) = 8$. Since on the heterotic side there are no instantons turned on, one expects no gauge bundle parameters, but a surviving gauge group from the unbroken $N = 2$ gauge group $U(1)^{16}$, which gives (in the C model, say) a rank 8 group; furthermore the greater rigidity on the F -theory side ($\#def K3_{12-8} = 20 - \rho = 10$ only instead of 18) translates itself to a corresponding rigidity of the $CY^{19,19}$ freezing also 8 moduli, i.e. leaving only $c = 38 - 8 = 30$ moduli there.

3 The Dual Superpotentials

3.1 F -theory superpotential

Recall that the authors of [12] find for the F -theory on X^4 a superpotential which is represented in the type IIB language by wrapping three branes over the four cycles of the form $C \times P_z^1$, where C is a rational curve in the del Pezzo of selfintersection $C^2 = -1$

(a condition being equivalent for a rational C on dP to $C^2 < 0$ and furthermore to being a section of the elliptic fibration), i.e.

$$W = \sum_{C^2=-1, C \text{ rational}} e^{2\pi i \langle c(C), z \rangle} \quad (3.12)$$

up to a prefactor, common to all divisors, with dependence on the complex structure moduli (there are further possibilities as well [23],[14]). Here $c(C)$ denotes the homology class of C , say expressed as⁶ $c_0 F + \sum_{i=1}^9 c_i E_i$ where F is the elliptic fibre of del Pezzo and the E_i are the nine blown up intersection points of two cubics in the projective plane; $z = (z_i)_{i=0,\dots,9}$ is a corresponding ten parameter vector in the dual cohomology, i.e. $\langle c(C), z \rangle = \sum_{i=0}^9 c_i z_i$. If one changes to base systems adapted to the E_8 intersection lattice⁷ one has with $z_0 := \tau$ for such a C that $\langle c(C), z \rangle = c_0 z_0 + \sum_{i=1}^9 c_i z_i = (m^2 + \frac{m_8}{3})\tau + \sum_{i=1}^8 m_i w_i + (z_9 - \frac{m_8}{3}\tau) = m^2\tau + (m, w) + z_9$, so ($q_9 = e^{2\pi i z_9}$):

$$W = q_9 \Theta_{E_8}(\tau, w_i) = q_9 \sum_{m \in \mathbf{Z}_{E_8}^8} q_\tau^{m^2} \prod_{i=1}^8 q_{w_i}^{m_i} \sim q_9 \sum_{a=1}^4 \prod_{i=1}^8 \theta_a(\tau, w_i), \quad (3.13)$$

being of modular weight 4 with respect to $PSL(2, Z)_\tau$ (the w_i transform as $w_i \rightarrow \frac{w_i}{c\tau+d}$). This superpotential is common to our models. Minimizing this superpotential leads to a supersymmetry preserving locus (essentially unique, i.e. up to the action of the Weyl group of E_8) consisting in locking pairs of the w_i on the four half-periods of the elliptic curve E_τ . Expanding in $\phi_i = w_i - w_i^0$ around the minima w_i^0 gives $W|_{SUSY} \sim q_9 \theta_1^2(\tau, \phi) \eta(\tau)^6$ behaving in leading order as $q_9 \phi^2 \eta(\tau)^{12}$ (for notational simplicity we have identified all ϕ_i).

Now we will give some arguments that the superpotential eq.(3.13) has to be corrected by a modular function. In fact, the authors of [12] expect that this expression for the superpotential has to be corrected by an η^8 denominator - leading to a completely modular invariant superpotential - when taking into account a correct counting of the sum of rational (-1)-curves including also reducible objects. We would like to argue that a different correction factor is required to get the correct modular weight for W , namely a factor $\eta(\tau)^{-12}$. Note that then the corrected superpotential $W' = W/\eta^{12}$ of modular weight -2 is around the minima w_i^0 simply given by a τ -independent mass term (μ -term) for the fields ϕ

$$W'|_{SUSY} \sim q_9 \phi^2. \quad (3.14)$$

To argue for this correction by η^{-12} we can compare with a precise rational curve counting on the del Pezzo provided by mirror symmetry [24]. In the $CY^{3,243}$ over F_1 (where dP occurs over the exceptional section of the F_1 base) one finds among the instanton numbers

$$\sum n_{0,1,k} q^k = \frac{E_4}{q^{-1/2} \eta^{12}}, \quad (3.15)$$

⁶Here by abuse of language we do not distinguish between the curves and their homology classes

⁷Essentially $A_i = E_i - E_{i+1}$, where a C then has coordinates m_i , say, instead of the n_i , and $w_i = z_i - z_{i+1}$, $i = 1, \dots, 7$; cf. [12] for details; z_9 is the Kähler modulus of the the base P_y^1 ; τ is the Kähler modulus of the fibre T_x^1 , and the w_i correspond to the E_8 part; note that $m^2 := \sum_{i=1}^8 m_i^2 - (m_1 m_2 + \dots + m_6 m_7 + m_3 m_8)$ is $-\frac{1}{2}$ times the \vec{m}^2 , say, built with the intersectionform of the (negative definite) E_8 intersection lattice

where the zero index indicates that we are considering the del Pezzo sector of the threefold and the 1 indicates that inside the del Pezzo itself one has $C \cdot f = 1$ as intersection with the elliptic fibre of del Pezzo; i.e. considering the $n_{0,1,k}$ sector implements exactly the counting we want to do. Now in the del Pezzo lattice $H \oplus E_8 = b\mathbf{C} \oplus f\mathbf{C} \oplus E_8$ (after suitable base change; $b^2 = -1, f^2 = 0, b \cdot f = 1$) one has for such a $C = \alpha b + \beta f + \vec{l} \cdot \vec{e}$ with $\alpha = 1, \beta = k$ that $-2 = C^2 + C \cdot K = C^2 - C \cdot f = -1 + 2\beta + \vec{l}^2 - 1$, i.e. $\beta = -\frac{1}{2}\vec{l}^2$ showing the E_4 of the naive count. So if we compare with the superpotential having all w_i locked to zero, which is given by $W(\tau, w_i = 0) = q_9 E_4(\tau)$, we find the asserted correction factor.

Note that the factor η^{-12} furthermore comes up not only also in a η^{-x} computation for del Pezzo, but even in⁸

$$e^{b\chi + c\sigma} = [(u^2 - 1) \frac{d\tau}{du}]^{\chi/4} (u^2 - 1)^{\sigma/8} = \left(\frac{2}{\pi i}\right)^3 \frac{1}{\theta_3^{12}} \frac{-\theta_3^8}{4\theta_2^4 \theta_4^4} = -\left(\frac{1}{2\pi i}\right)^3 \frac{1}{\eta^{12}} \quad (3.16)$$

occurring in connection with the question of integration over the u -plane [26] .

3.2 Heterotic Superpotential

Now according to [11] the superpotential generating divisors on the F-theory side correspond in our case to world-sheet instantons on the heterotic side. We want all the world-sheet-instantons/rational curves to contribute to get a match with F -theory. So either we should not have a nontrivial bundle embedded at all, i.e. we should reach our situation in the bundle sector in models C, C' , i.e. from an $(0, 0; 24)$ startpoint (anomaly cancellation purely by fivebranes) in $N = 2$; so in this case the rational curves are not obstructed at all to contribute to the superpotential as the spin bundle $\mathcal{O}(-1)$ of a rational curve will not be tensored with an embedded bundle and so no fermion modes are created. Alternatively we could start from a nonstandard embedding (i.e. not purely in one E_8) like $(12, 12; 0)$ in $N = 2$, i.e A' model setup, where also all world-sheet-instantons may contribute to the superpotential.

The rational instanton numbers of the $CY^{19,19}$ are essentially determined by the dP geometry (for more details see appendix A.2). (Since the dP base is common to the F -theory fourfolds and to the heterotic $CY^{19,19}$ one can more or less immediately deduce the equality of the superpotentials.) So let us read of the (naively counted) rational instanton numbers of the $CY^{19,19}$:

$$n_{k_{\tau'}, (k_{w'_i}), k_9, k_{\tau}, (k_{w_i})} = \delta_{k_{\tau'}, (k_{w'_i})^2} \delta_{k_9, 1} \delta_{k_{\tau}, (k_{w_i})^2}. \quad (3.17)$$

These instanton numbers lead to the following heterotic superpotential

$$W = \Theta_{E_8}(\tau', w'_i) q_9 \Theta_{E_8}(\tau, w_i). \quad (3.18)$$

Note that the heterotic computation leads to a second E_8 theta-function, which should appear in the prefactor on the F -theory side.

⁸Namely $u = 1 - 2\frac{\theta_2^4}{\theta_3^4} = \frac{\theta_4^4 - \theta_2^4}{\theta_3^4}$ gives (using that $\theta_3^4 \text{ resp. } 2 = 8\frac{1}{2\pi i} \partial_{\tau} \log \frac{\theta_2}{\theta_4} \text{ resp. } 3$ (cf. [25]) so that $4\partial_{\tau} \log \frac{\theta_2}{\theta_3} = \pi i \theta_4^4$) that $\partial_{\tau} u = (u - 1) \partial_{\tau} \log(1 - u) = (u - 1) 4\partial_{\tau} \log \frac{\theta_2}{\theta_3} = (u - 1) \pi i \theta_4^4 = -2\pi i \frac{\theta_2^4 \theta_4^4}{\theta_3^4}$, so $\frac{\partial \tau}{\partial u} = -\frac{1}{2\pi i} \frac{\theta_3^4}{\theta_2^4 \theta_4^4}$; on the other hand $u^2 - 1 = \frac{(\theta_4^4 - \theta_2^4)^2 - \theta_3^8}{\theta_3^8} = -4 \frac{\theta_2^4 \theta_4^4}{\theta_3^8}$ and so $(u^2 - 1) \frac{\partial \tau}{\partial u} = \frac{4}{2\pi i} \frac{1}{\theta_3^4}$.

Let us now discuss the modular properties of the heterotic superpotential. If we follow the construction of the heterotic string on the Calabi-Yau $CY^{19,19}$ as an $\mathbf{Z}_2 \times \mathbf{Z}'_2$ orbifold (cf. A2), we can see that the perturbative heterotic superpotential, which describes a mass term for Wilson line moduli fields, is constrained by the unbroken target space duality symmetries of the orbifold in such a way that the superpotential, which includes the factor $\eta(\tau)^{-12}$, has the correct modular weight. For this consider the τ -dependent superpotential in the orbifold limit. In $N = 1$ supergravity the Kähler potential K and the superpotential W are connected, and the matter part of the $N = 1$ supergravity Lagrangian [27] is described by a single function $G(\phi, \bar{\phi}) = K(\phi, \bar{\phi}) + \log |W(\phi)|^2$, where the ϕ 's are chiral superfields. The target space duality transformations act as discrete reparametrization on the scalars ϕ and induce simultaneously a Kähler transformation on K . Invariance of the effective action constrains W to transform as a modular form of particular weight [17]; specifically under $PSL(2, Z)_{T_1} \times PSL(2, Z)_{T_3}$ the superpotential must have modular weights -1, i.e. it has to transform under $T_{1,3} \rightarrow \frac{a_{1,3}T_{1,3}+b_{1,3}}{c_{1,3}T_{1,3}+d_{1,3}}$ as $W \rightarrow \frac{W}{(c_1T_1+d_1)(c_3T_3+d_3)}$; in particular the μ -term

$$W = \phi_{1,i}\phi_{3,i} \quad (3.19)$$

has the required modular weight and precisely matches with $W'|_{SUSY}$ in eq.(3.14) (a more general form would be given by $W = \frac{\phi_{1,i}^{l_1}\phi_{3,i}^{l_3}}{\eta(T_1)^{-2l_1+2}\eta(T_3)^{-2l_3+2}}$). Now, as explained in appendix A2, the Calabi-Yau Kähler modulus τ corresponds in the orbifold limit to the diagonal deformation $\tau = T_1 = T_3$. Then, concerning the transformation properties of the superpotential under the diagonal modular transformations $PSL(2, Z)_\tau$, invariance of the G -function requires that W has modular weight -2, i.e. that under $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ one has $W \rightarrow \frac{W}{(c\tau+d)^2}$. Clearly the mentioned μ -term has the correct modular weight (a more general function of τ and ϕ_i is given by $W = \frac{\phi_i^n}{\eta(\tau)^{-2n+4}}$). As already discussed, the superpotential (3.19) has the supersymmetry preserving minimum ($W = 0$, $dW = 0$) $\phi_{1,i} = \phi_{3,i} = 0$. Therefore the vacuum expectation values of the Wilson line fields $\phi_{1,i}$, $\phi_{3,i}$ are set to zero after the minimization, i.e. the vacuum expectation values are not free, continuous parameters in the presence of this superpotential. Going away from the minimum of the superpotential by turning on the Wilson line fields $\phi_{1,i}$, $\phi_{3,i}$ means in the context of conformal field theory, that one is in fact going away from the conformal point, i.e. going off-shell.

Finally let us also remark on the factor q_9 in eq.(3.13). In the orbifold limit the possible z_9 -dependence of the superpotential is again restricted by T -duality. However the duality group with respect to the modulus z_9 is no longer the full modular group $PSL(2, Z)$ but only a subgroup of it, since the $R \rightarrow 1/R$ duality is broken by the freely acting \mathbf{Z}_2 in this sector (the space P^1 has no $R \rightarrow 1/R$ duality due to the absence of winding modes in this sector): so the superpotential is not required to transform as a modular function, but it should be just a periodic function in $\Re z_9$, like q_9 . These kind of functions generically arise as the zero mode prefactor in the large z_9 limit (i.e. suppressing all winding modes in this decompactification limit) of modular functions, like the η -function or the θ -functions. Just consider the following naive example. We can regard [28] the superpotential as the sum over the massive (BPS) spectrum of the orbifold compactification: $W \sim \prod M^{-1}$. For example, summing over all momentum and winding states of a two-torus compactification with masses $M = m + nT$ in a $SL(2, Z)_T$ invariant way yields

the T -dependent superpotential $W \sim \prod_{m,n} (m + nT) \sim \eta(T)^{-2}$, which has the required modular weight -1. Similarly summing over all momentum and winding modes of the shifted lattice, corresponding to the free plane z_2 , one obtains⁹ a factor $W \sim \theta_2(T_2)^{-2}$, leaving one in the limit of large $\Im T_2$, i.e. suppressing all winding modes in this sum, with the zero mode piece: $W \sim e^{-2\pi i T_2/4} = e^{-2\pi i z_9} = q_9^{-1}$.

In summary we have supported some strong evidence that the perturbative heterotic superpotential matches with its F -theory counterpart. It would be very interesting to analyze models with (modular) non-perturbative, S -dependent heterotic superpotentials [29] and their F -theory duals. In this context non-perturbative symmetries in underlying $N = 2$ models, like the S - T exchange symmetry in the $CY^{3,243}$, may play an important role.

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A Appendix

A.1 The del Pezzo surface

The representation $\left[\begin{smallmatrix} P_x^2 \\ P_y^1 \end{smallmatrix} \middle| \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right]$ of the del Pezzo makes visible on the one hand its elliptic fibration over P_y^1 via the projection onto the second factor; on the other hand the defining equation $C(x_0, x_1, x_2)y_0 + C'(x_0, x_1, x_2)y_1 = 0$ shows that the projection onto the first factor exhibits dP as being a P_x^2 blown up in 9 points (of $C \cap C'$), so having as nontrivial hodge number (besides b_0, b_4) only $h^{1,1} = 1 + 9$. Furthermore the dP has 8 complex structure moduli: they can be seen as the parameter input in the construction of blowing up the plane in the 9 intersection points of two cubics (the ninth of which is then always already determined as they sum up to zero in the addition law on the elliptic curve; so one ends up with $8 \times 2 - 8$ parameters) or - counting via number of inequivalent monomials - as $10 \cdot 2 - (8 + 3) - 1$.

The dP can be obtained from $K3$ by a \mathbf{Z}_2 modding. This corresponds to having on $K3$ a Nikulin involution of type (10,8,0) with two fixed elliptic fibers in the $K3$ leading to

$$\begin{array}{ccc} K3 & \rightarrow & dP \\ \downarrow & & \downarrow \\ P_y^1 & \rightarrow & P_{\tilde{y}}^1 \end{array} \quad (A.20)$$

induced from the quadratic base map $y \rightarrow \tilde{y} := y^2$ with the two branch points 0 and ∞ (being the identity along the fibers). One can follow this relation also in the orbifold representation of $K3$ as T^4/Z_2 , where the involution operates on the T^2 's as sign-flip; this shows also the fibration by the first, say, T^2 over the $P^1 = T^2/Z_2$ coming from the second T^2 in a double covering having 4 branch points leading to four $\tilde{D}_4 = I_0^*$ fibers.

⁹We thank C. Kounnas for discussion on this point.

Now (del Pezzo being K3 divided by an involution having two fixed fibers) do a second \mathbf{Z}_2 modding given by an involution of the base coordinate y together with an half lattice shift $1/2$: $(x, y) \rightarrow (x, -y + 1/2)$. This destroys essentially half of the cohomology of K3 leading to $H \oplus E_8$ as intersection form of dP .

In the Weierstrass representation $y^2 = x^3 - f_8(u)x - g_{12}(u)$ of K3 the mentioned quadratic redefinition translates to the representation $y^2 = x^3 - f_4(u)x - g_6(u)$ of dP (showing again the $8 = 5 + 7 - 3 - 1$ deformations). Repeated use will be made in the paper of the fact that del Pezzo can be reached, in the sense of turning on complex deformations, from the \mathbf{Z}_2 modded (via the mentioned quadratic base map, now with sign-flip in the fibers) *constant* elliptic fibration over P^1

$$\begin{array}{ccc} P^1 \times T^2 & \rightarrow & X_{11}(j) \\ \downarrow & & \downarrow \\ P^1 & \rightarrow & P^1 \end{array} \quad . \quad (\text{A.21})$$

Here $X_{11}(j)$ is the (almost) constant fibration of elliptic curve of invariant j with two $\tilde{D}_4 = I_0^*$ singular fibers over the two branch points of the quadratic base map (cf. [30]). One finds this degenerate del Pezzo on the boundary of the complex structure moduli space at $f_4 = ru^2, g_6 = su^3$: $j = \frac{4r^3}{4r^3 + 27s^2}$, the two singular fibers are at $u = 0, \infty \in P^1$ and one sees the four sections (in the covering above given by the constant fibration exactly the sections given by the halfdivision points of the T^2 survive the modding): they are $(x, y) = (x_i u, 0)$, where x_i solves $x^3 + rx + s = 0$, besides the the zero section given by the point at infinity.

A.2 The Calabi-Yau $CY^{19,19}$

The $CY^{(19,19)} = \left[\begin{array}{c} P^2 \\ P^1 \\ P^2 \end{array} \left| \begin{array}{cc} 3 & 0 \\ 1 & 1 \\ 0 & 3 \end{array} \right. \right] = dP \times_{P_y^1} dP$, which is elliptically fibered over del Pezzo, can be obtained from $T^2 \times K3 = \left[\begin{array}{c} P^2 \\ P^1 \\ P^2 \end{array} \left| \begin{array}{cc} 3 & 0 \\ 0 & 2 \\ 0 & 3 \end{array} \right. \right]$ by the Voisin-Borcea involution, which consists in the 'del Pezzo' involution (type (10,8,0) in Nikulin's classification) with two fixed elliptic fibers in the K3 combined with the usual "-"-involution with four fixed points in the T^2 ; this leads in the base to the relation mentioned in A.1 and in the $P_y^1 \times T^2$ 'plane' to the X_{11} mentioned in A.1 and so to the second del Pezzo. So here the symmetric 'degree one' entries in the P^1 variables have a seemingly different origin: one by '*reduction*' (from two) and one by '*emergence*' (from zero). There is of course only an apparent asymmetry in the situation: the fibration in the $P_y^1 \times T^2$ 'plane' with the two singular $\tilde{D}_4 = I_0^*$ fibers is \mathbf{Z}_2 covered by the orbifold limit of K3 (instead of the smooth $K3 = \left[\begin{array}{c} P^1 \\ P^2 \end{array} \left| \begin{array}{c} 2 \\ 3 \end{array} \right. \right]$) with four $\tilde{D}_4 = I_0^*$ fibers, i.e. a fully symmetric start could be done from $T^6/(\mathbf{Z}_2 \times \mathbf{Z}_2)$. So, to elaborate on the construction of the heterotic string on the $CY^{19,19}$ as an $\mathbf{Z}_2 \times \mathbf{Z}_2'$ orbifold, consider first the heterotic string on the six-torus $T_1^2 \otimes T_2^2 \otimes T_3^2$ with Kähler moduli T_j ($j = 1, 2, 3$), and complex structure moduli U_j . The $\mathbf{Z}_2 \times \mathbf{Z}_2'$ acts on the three complex coordinates (z_1, z_2, z_3) as

$$\begin{aligned} \alpha : \quad & (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3), \\ \beta : \quad & (z_1, z_2, z_3) \rightarrow (-z_1, z_2 + \frac{1}{2}, -z_3), \end{aligned}$$

$$\alpha\beta : (z_1, z_2, z_3) \rightarrow (z_1, -z_2 + \frac{1}{2}, -z_3). \quad (\text{A.22})$$

These three \mathbf{Z}_2 moddings define three $N = 2$ subsectors in the total Hilbert space. The first α -modding acts non-freely; this modding corresponds to the orbifold limit of the heterotic string on $K3_{1,2} \times T_3^2$. In the same way, the non-free $\alpha\beta$ -modding corresponds to a $K3_{23} \times T_1^2$ compactification. So the situation is symmetric with respect to the α - and $\alpha\beta$ -modding thus reflecting the fibre product structure of the $CY^{19,19}$. On the other hand, the generator β acts freely on the original six-torus, so we call z_2 the free plane. Let us relate now the moduli of the orbifold compactification on $(T_1^2 \otimes T_2^2 \otimes T_3^2)/(\mathbf{Z}_2 \times \mathbf{Z}_2)$ and the Calabi-Yau moduli. First compare the moduli T_j with the three Calabi-Yau moduli z_9 , τ and τ' . The free plane z_2 corresponds to the P^1 base of dP , so $4z_9 = T_2$, where the factor 4 arises because the volume is reduced two times by half going from T^2 to P_{K3}^1 to P_{dP}^1 . On the other hand, since the $K3 \times T^2$ compactification is obtained by the α -modding as well as by the $\alpha\beta$ -modding, the modulus T_1 does not correspond, say, to the modulus τ of the elliptic fibre of dP , but the moduli τ and τ' correspond to certain linear combinations of orbifold states: τ , say, to the deformation along the diagonal $\tau = T_1 = T_3$, τ' to the orthogonal deformation (the symmetry between the α - and $\alpha\beta$ -modding enforces us to take these linear combinations). The same type of identification holds for the other moduli like the w_i of the $CY^{19,19}$ in terms of again identified orbifold Wilson line fields, i.e. $w_i - w_i^0 = \phi_i = \phi_{1,i} = \phi_{3,i}$, where the $\phi_{1,i}$, $\phi_{3,i}$ ($i = 1, \dots, n_1 = n_3 = 8$) belong to the first resp. third torus. The classical Kähler potential of the fields T_i , U_i , $\phi_{1,i}$ and $\phi_{3,i}$ has the form $K = -\sum_{j=1}^3 \log[(T_j - \bar{T}_j)(U_j - \bar{U}_j) - \sum_{i=1, j \neq 2}^{n_j} (\phi_{j,i} + \bar{\phi}_{j,i})^2]$. The unbroken T-duality group contains $PSL(2, Z)_{T_1} \times PSL(2, Z)_{T_3}$, which acts as $T_{1,3} \rightarrow \frac{a_{1,3}T_{1,3} + b_{1,3}}{c_{1,3}T_{1,3} + d_{1,3}}$, $\phi_{1,i} \rightarrow \frac{\phi_{1,i}}{c_1T_1 + d_1}$, $\phi_{3,i} \rightarrow \frac{\phi_{3,i}}{c_3T_3 + d_3}$. Hence under simultaneous modular transformations ($a = a_1 = a_3$, etc.) along the diagonal, $\tau = T_1 = T_3$, the group $PSL(2, Z)_\tau$ has the action $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$, $\phi_i \rightarrow \frac{\phi_i}{c\tau + d}$.

Let us now consider the rational curves in the $CY^{(19,19)}$. Let $\mathcal{O}(2) \oplus \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$ be the splitting type of (the tangential bundle of CY over) the rational curve C. Then $\mathcal{O}(2) \oplus \mathcal{O}(a_i)$ is the corresponding splitting type of the projected rational curve C_i of normal bundle $\mathcal{O}(C_i^2)$ in the del Pezzos dP_i ($i = 1, 2$). From $-e_{C_i} = C_i^2 + C_i \cdot K_{dP_i} = a_i - C_i \cdot f_i$ you see that $a_i \geq -1$ (cf. also [11]) which together with $a_1 + a_2 = -2$ shows $a_i = -1$ (in other words: *all* rational curves lying in $CY^{19,19}$ project in the dP_i factors to the *special* rational curves of selfintersection -1). That is we get for the (naively read of) instanton numbers of the $CY^{(19,19)}$ (cf. sect. 3.1 and [12])

$$n_{k_{\tau'}, (k_{w_i'}), k_9, k_\tau, (k_{w_i})} = \delta_{k_{\tau'}, (k_{w_i'})^2} \delta_{k_9, 1} \delta_{k_\tau, (k_{w_i})^2}. \quad (\text{A.23})$$

To rephrase the process of constructing rational curves: in the beginning one has the base P_y^1 ; then one embeds it in the del Pezzo base as section; then to get really a curve in the threefold represented by the fibered product of the two del Pezzos one has to do the same process also for the other fibration direction giving the symmetrical result indicated; as the processes are - besides the common base - independent, one gets the second factor.

For use in the next section let us point out the existence of a conifold transition to the CY $\left[\begin{smallmatrix} P^2 \\ P^2 \end{smallmatrix} \middle| \begin{smallmatrix} 3 \\ 3 \end{smallmatrix} \right]$ of $h^{(1,1)} = 2$ and $h^{(2,1)} = 10 \cdot 10 - (8 + 8) - 1 = 83$ and so of Euler number -162. If you start from $CY^{(19,19)} = \left[\begin{smallmatrix} P^2 \\ P_y^1 \\ P_b^2 \end{smallmatrix} \middle| \begin{smallmatrix} 3 & 0 \\ 1 & 1 \\ 0 & 3 \end{smallmatrix} \right]$ with the equations $y_0 Q_a + y_1 R_a = 0$

and $y_0 S_b + y_1 T_b = 0$ one sees that the existence condition for y gives the special bicubic $Q_a T_b - R_a S_b = 0$ and the singular set (for it) $Q = R = S = T = 0$ of 81 nodes; i.e. contract in the $CY^{(19,19)}$ say 81 P^1 's (coming from combining respectively 9 sections in each del Pezzo of the fibre product) and then deform to a generic bicubic (i.e. detune $83 - 19 = 64 = 8 \cdot 8 = (9 - 1) \cdot (9 - 1)$ parameters).

Note that if J_i ($i = 1, 2, 3$) denote the induced classes from the factors in $CY^{(19,19)} = \left[\begin{smallmatrix} P^2 \\ P^1 \\ P^2 \end{smallmatrix} \middle| \begin{smallmatrix} 3 & 0 \\ 1 & 1 \\ 0 & 3 \end{smallmatrix} \right]$, m_a the dimensions of the factor spaces and d_i^a ($i = 1, 2$) the respective degrees of the two defining equations one has $c_2^{ab} = \frac{1}{2}(-\delta^{ab}(m_a + 1) + \sum_{i=1}^2 d_i^a d_i^b)$, so $c_2 = 3(J_1^2 + J_3^2 + J_2(J_1 + J_3))$. Furthermore one has (as $dP = \left[\begin{smallmatrix} P^2 \\ P^1 \end{smallmatrix} \middle| \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right]$ has intersection form $\begin{pmatrix} 1 & 3 \\ 3 & 0 \end{pmatrix}$ for the divisors (line resp. point) induced from the factors) for the intersection numbers of the CY that $K^0 = 3J_1^2 J_3 + 3J_1 J_3^2 + 9J_1 J_2 J_3$.

A.3 The Calabi-Yau four-fold X_A^4

Remember that 4-fold X^4 (model A) of section (2.1) could be represented as the fibre product $X^4 = dP \times_{P_y^1} \mathcal{B}$, where $\mathcal{B} = \left[\begin{smallmatrix} P_y^1 \\ P_y^1 \\ P^2 \end{smallmatrix} \middle| \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right]$. This gives $h^{(1,1)}(X^4) = 12$: the $10 + 1$ classes of $dP \times P_z^1$ plus the elliptic fibre class of F-theory ($\rho_{K3_{12-8}} = 2$). Next the number of complex deformations of $X^4 = dP \times_{P_y^1} \mathcal{B}$ can be counted as the sum of the number of complex deformations of dP and \mathcal{B} plus 3 (because you can use only one times the reparametrization freedom of the common base; you can compare that procedure with the count in the similar case of the $CY^{(19,19)}$, where you can count $19 = 8 + 8 + 3$). Now the deformations of \mathcal{B} (which is here not the $h^{(2,1)}$ as we are not on a CY) are counted as $2 \cdot 3 \cdot 10 - (3 + 3 + 8) - 1 = 45$ giving $h^{3,1}(X^4) = 45 + 8 + 3 = 56$ and with $h^{1,1} + h^{3,1} - h^{2,1} = \frac{x}{6} - 8 = 40$ of [18] also $h^{2,1} = h^{3,1} - 28 = 28$.¹⁰

To gain further confidence in $h^{(2,1)}(\mathcal{B}') = 75$, where $\mathcal{B}' = \left[\begin{smallmatrix} P^1 \\ P^1 \\ P^2 \end{smallmatrix} \middle| \begin{smallmatrix} 2 \\ 2 \\ 3 \end{smallmatrix} \right]$, note that not only the Euler number can be independently computed from the degrees to be -144, leading to the 75, but that (cf. [20]) one can follow the precise occurrence of that CY through a conifold transition from (remarkably enough again our friend) $CY^{(19,19)}$. To see this observe that in contrast to the easier case (considered in A.2) of transition to the CY $\left[\begin{smallmatrix} P^2 \\ P^2 \end{smallmatrix} \middle| \begin{smallmatrix} 3 \\ 3 \end{smallmatrix} \right]$ in our case here one is choosing in one del Pezzo only 8 sections leading to the contraction of $8 \cdot 9 = 72$ P^1 's and then detuning of $75 - 19 = 56 = 7 \cdot 8 = (8 - 1) \cdot (9 - 1)$ parameters; i.e. the Euler number -144 is reached (from the Euler number zero of the $CY^{(19,19)}$) in the usual two steps: first the contraction of the 72 P^1 's gives $\chi = -72$ and then the resmoothing via introduction of the three-spheres lets it go to -144. (To imple-

¹⁰ One can check explicitly that $h^{2,1} = h^{2,1}(\mathcal{B})$. One can compute directly from the given degrees that $e_{\mathcal{B}} = -48$ resp. $e_{\mathcal{B}'} = -144$, which matches with the visualization of \mathcal{B}' as branched covering of \mathcal{B} (induced from a two-fold covering with two branch points of the base P_y^1), namely $-144 = 2(-48) - 2(24)$, where one sees that the two fixed fibers over the two fixed points in the base are now K3's. Now $e_{\mathcal{B}} = -48$ gives with $h^{(1,0)}(\mathcal{B}) = h^{(2,0)}(\mathcal{B}) = h^{(3,0)}(\mathcal{B}) = 0$ (cf. [12]) and $-48 = 2 + 2(h^{(1,1)} - h^{(2,1)})$ that $h^{2,1}(\mathcal{B}) = 28 = h^{2,1}$, which is also a number of quite visible origin: \mathcal{B} can be considered (cf.[12]) as the blow-up of $\left[\begin{smallmatrix} P^1 \\ P^2 \end{smallmatrix} \right]$ at the base locus $\Gamma := \left[\begin{smallmatrix} P^1 \\ P^2 \end{smallmatrix} \middle| \begin{smallmatrix} 2 & 2 \\ 3 & 3 \end{smallmatrix} \right]$ of the pencil of K3 surfaces. The cohomology class of Γ is $-2J_1 - 3J_2$, denoting by J_i the respective classes coming from the factors, so $e_{\Gamma} = -\Gamma^2 = -54$ (showing again $e_{\mathcal{B}} = 6 - 54 = -48$), i.e. Γ has genus 28.

ment the special features of this transition it is useful, after replacing one elliptic fibre of the $CY^{19,19}$ by $\left[\begin{array}{c|c} P^1 & 2 \\ \hline P^1 & 2 \end{array} \right]$, to consider also then the $\left[\begin{array}{c|c} P^1 & 2 \\ \hline P^1 & 1 \\ \hline P^2 & 0 \end{array} \right]$, where now $\left[\begin{array}{c|c} P^1 & 2 \\ \hline P^1 & 2 \end{array} \right]$ consists of 8 points instead of the 9 of $\left[\begin{array}{c|c} P^2 & 3 \end{array} \right]$.)

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